Tutorial 12.

① Cashflow Duration

Series of payments $K_1, K_2, \cdots, K_n$, the present value $P = \sum_{t=1}^{n} k_t(1+i)^{-t}$, $\frac{dP}{di} = -\sum_{t=1}^{n} t k_t (1+i)^{-t-1}$.

the modified duration is $DM = \frac{-\frac{dP}{di}}{P} = \frac{\sum_{t=1}^{n} t K_t (1+i)^{-t-1}}{\sum_{t=1}^{n} K_t (1+i)^{-t}}$ and Macaulay duration is

$$D = DM(1+i) = \frac{\sum_{t=1}^{n} t k_t (1+i)^{-t}}{\sum_{t=1}^{n} k_t (1+i)^{-t}}.$$

For coupon bond, $K_1 = \cdots = k_{n-1} = Fr$, $k_n = F(1+r)$, then

$$D = \frac{\sum_{t=1}^{n-1} t Fr(1+i)^{-t} + nF(1+r)(1+i)^{-n}}{\sum_{t=1}^{n-1} Fr(1+i)^{-t} + F(1+r)(1+i)^{-n}}$$

For portfolio, the Macaulay duration is $D = -(1+i)\frac{\frac{dX}{di}}{X} = \frac{\sum_{k=1}^{n} -(1+i)\frac{dX_k}{di}}{X} = \sum_{k=1}^{n} \frac{D_k \cdot X_k}{X}$

$X = X_1 + \cdots + X_n$ in cashflows, $D_k = -(1+i)\frac{\frac{dX_k}{di}}{X_k}$

7.1.6.

Bond 1: $D_1 = 12.7$, $X_1 = \frac{88.35}{100} F_1$, Bond 2: $D_2 = 14.6$, $X_2 = \frac{130.49}{100} F_2$.

Macaulay duration $D = \sum_{k=1}^{n} \frac{D_k X_k}{X} = \frac{D_1 X_1 + D_2 X_2}{X_1 + X_2} = \frac{12.7 \times \frac{88.35}{100} F_1 + 14.6 \times \frac{130.49}{100} F_2}{\frac{88.35}{100} F_1 + \frac{130.49}{100}} = 13.5.$

We also have $F_1 + F_2 = 100$, then $F_1 = 67.01$, $F_2 = 32.99$

$X = \frac{88.35}{100} \times 67.01 + \frac{130.49}{100} \times 32.99 = 102.25.$

② Immunization.

$(PV_A(i_0) = \sum A_t v_{i_0}^t) = (PV_L(i_0) = \sum L_t v_{i_0}^t)$

Our expectation is to make $PV_A(i) > PV_L(i)$ when $i \to i_0$

if $PV_A(i_0) = PV_L(i_0)$, $\frac{d\,PV_A(i)}{di}\Big|_{i=i_0} = \frac{d\,PV_L(i)}{di}\Big|_{i=i_0}$, $\frac{d^2 PV_A(i)}{di^2}\Big|_{i=i_0} > \frac{d^2 PV_L(i)}{di^2}\Big|_{i=i_0}$ then the liability cashflows are Redington immunized by asset cash flows.

7.2.7

(a) $PV_A(i_0) = PV_L(i_0) \Rightarrow \frac{A_1}{1.1} + \frac{A_5}{1.1^5} = 100\left(\frac{1}{1.1^2} + \frac{1}{1.1^4} + \frac{1}{1.1^6}\right) = 207.39$

$\frac{d\,PV_A(i)}{di}\Big|_{i=i_0} = \frac{dPV_L(i)}{di}\Big|_{i=i_0} \Rightarrow -\frac{A_1}{1.1^2} - \frac{5A_5}{1.1^6} = 100\left(-\frac{2}{1.1^3} - \frac{4}{1.1^5} - \frac{6}{1.1^7}\right) \Rightarrow \frac{A_1}{1.1} + \frac{5A_5}{1.1^5} = 777.18,$

$\Rightarrow A_1 = 71.44,\ A_1 = 229.41$

(b) $\frac{d^2}{di^2} PV_A(i)\Big|_{i_0} > \frac{d^2}{di^2} PV_L(i)\Big|_{i_0}$

$\Rightarrow \frac{2A_1}{1.1} + \frac{6 \times 5A_1}{(1.1)^5} = 4403 > 100\left(\frac{2 \times 3}{1.1^2} + \frac{5 \times 4}{1.1^4} + \frac{6 \times 5}{1.1^6}\right) = 3775.$ It's Redington immunization.

③ Forward Contracts

Non-arbitrage forward price $K = S_0 e^{rT}$ $\quad r$: risk-free rate.

At time t, the value of forward contract is $\underbrace{(S_T - K)}_{\text{final value of forward contract}} e^{-r(T-t)} = S_t - Ke^{-r(T-t)}$

$= S_t - S_0 e^{rt}$.

9.1.2.

(a) $K_1 = S_0 e^{rT_1} = 900 e^{0.08\times 1} = 974.96$, $\quad K_2 = S_0 e^{rT_2} = 900 e^{0.08\times 2} = 1056.16$.

(b) At time 1, $t=1$. 2-year contract value ($T=2$): $(S_T - K)e^{-r(T-t)} = S_t - S_0 e^{rt} = S_1 - 900 e^{0.08\times 1}$

3-year contract value ($T=3$): $-(S_T - K)e^{-r(T-t)} = Ke^{-r(T-t)} - S_T e^{-r(T-t)}$

$= S_0 e^{rt} - S_t = 900 e^{0.08\times 1} - S_1$

the combined value is 0.

(c) At $t=1$, $\quad T=2$: $(S_1 - K)e^{-r'(T-t)} = S_1 - Ke^{-r'(T-t)} = S_1 - S_0 e^{rT - r'T + r't}$

$r = 8\%$, $r' = 10\%$. $\quad = S_1 - 900 e^{(r-r')T + r't} = S_1 - 900 e^{-0.02\times 2 + 0.1}$

$= S_1 - 900 e^{0.06}$

$T=3$: $(K - S_T)e^{-r'(T-t)} = S_0 e^{rT} e^{-r'(T-t)} - S_1$

$= 900 e^{(r-r')T + r't} - S_1$

$= 900 e^{-0.02\times 3 + 0.1} - S_1 = 900 e^{0.04} - S_1$

the combined value is $\quad 900 e^{0.04} - 900 e^{0.06} = -18.92$

Problem Set 11: 7.1.1, 7.1.6, 7.2.1, 7.2.2, 7.2.7, 9.1.1, 9.1.2.

Tutorial: 7.1.6, 7.2.7, 9.1.2.

7.1.1

Macauley duration: $D = (1+i) \cdot DM = (1+i) \cdot \frac{-\frac{d}{di}P}{P} = \frac{\sum_{t=1}^{n} t K_t (1+i)^{-t}}{\sum_{t=1}^{n} K_t (1+i)^{-t}}$,

since $K_t = Fr$ for $t = 1, 2, \cdots, n-1$, $K_n = F + Fr$, then $Fr = 10$, $n = 3$, $j = 11.8\%$, $F = 100$

$$D = \frac{\sum_{t=1}^{n} t\, Fr\, v_j^t + n F v_j^n}{\sum_{t=1}^{n} Fr\, v_j^t + F v_j^n} = \frac{3 \times 100 \times (\frac{1}{1.118})^3 + 10[\frac{1}{1.118} + \frac{2}{1.118^2} + \frac{3}{1.118^3}]}{100 \times \frac{1}{1.118^3} + 10[\frac{1}{1.118} + \frac{1}{1.118^2} + \frac{1}{1.118^3}]}$$

$$= \frac{261.1}{95.7} = 2.73$$

7.1.6.

$P_1 = 88.35$, $P_2 = 130.49$, ~~$F_1 = 100$~~ ~~$F_2 = 100$~~. $D_1 = 12.7$, $D_2 = 14.6$.

$D = 13.5$.

The Macaulay duration of the portfolio is

$$D = \frac{\sum_{k=1}^{2} D_k \cdot X_k}{X} \Rightarrow \frac{\frac{88.35}{100} \times F_1 \times 12.7 + \frac{130.49}{100} \times F_2 \times 14.6}{\frac{88.35}{100} \times F_1 + \frac{130.49}{100} \times F_2} = 13.5, \text{ and } F_1 + F_2 = 100.$$

$\Rightarrow F_1 = 67.01$, $F_2 = 32.99$

X (Portfolio value) $= F_1 \cdot \frac{88.35}{100} + F_2 \cdot \frac{130.49}{100} = 102.25$.

7.2.1.

(a) x_1 units of bonds (i), x_2 units of bonds (ii).

cashflow at time 1: $1.01 x_1 + 0.02 x_2$

at time 2: $1.02 x_2$

To cover the liabilities of 1 in time 1 and time 2, we should guarantee

$1.01 x_1 + 0.02 x_2 = 1$, $1.02 x_2 = 1$, $\Rightarrow x_1 = 0.970685$, $x_2 = 0.980392$.

the present value is $x_1 \cdot \frac{C_1^{i}}{1+j_1} + x_2 (\frac{C_1^{ii}}{1+j_2} + \frac{C_2^{ii}}{(1+j_2)^2}) = 0.97 \times \frac{1.01}{1.14} + 0.98 \times (\frac{0.02}{1.15} + \frac{1.02}{1.15^2}) = 1.633187$.

(b) x_1 units of bond (i). x_3 units of bond (iii), $1.01 x_1 + 0.2 x_2 = 1$, $1.2 x_2 = 1 \Rightarrow x_1 = 0.825083$, $x_2 = 0.8333$

the present value is $0.83 \times \frac{1.01}{1.14} + 0.833 \times (\frac{0.2}{1.1495} + \frac{1.2}{1.1495^2}) = 1.632786$.

(c) impossible for (ii) and (iii) then compare (a) and (b), (b) is better.

7.2.2.

$PV_L(0.10) = 300{,}000$, $n=5$. $X \cdot a_{\overline{5}|a|} = 300{,}000 \Rightarrow X = 79{,}139$. annual payments

$n=15$, $X \cdot a_{\overline{15}|a|} = 300{,}000 \Rightarrow X = 39{,}442$

$n=50$, $X \cdot a_{\overline{50}|a|} = 300{,}000 \Rightarrow X = 30{,}257$.

$n=100$, $X \cdot a_{\overline{100}|a|} = 300{,}000 \Rightarrow X = 30{,}002$.

7.2.7.

(a) Asset = Liability, present value $= \frac{A_1}{1.1} + \frac{A_5}{(1.1)^5} = 100\left(\frac{1}{1.1^2} + \frac{1}{1.1^4} + \frac{1}{1.1^6}\right) = 207.39$.

By $PV_A(i_0) = \sum_0^n A_t v_{i_0}^t = \sum_0^n L_t v_{i_0}^t = PV_L(i_0)$, $\sum_0^n (A_t - L_t)(1+i_0)^{n-t} = 0$.

$\frac{d}{di} PV_A(i)\Big|_{i_0} = \frac{d}{di} PV_L(i)\Big|_{i_0}$. then $-\frac{A_1}{(1.1)^2} - \frac{5A_5}{(1.1)^6} = 100\left(-\frac{2}{1.1^3} - \frac{4}{1.1^5} - \frac{6}{1.1^7}\right)$

$\Rightarrow \frac{A_1}{1.1} + \frac{5A_5}{1.1^5} = 777.18$

$\Rightarrow A_1 = 71.49$, $A_5 = 229.91$.

(b) The condition is $\frac{d^2}{di^2} PV_A(i)\Big|_{i_0} > \frac{d^2}{di^2} PV_L(i)\Big|_{i_0}$

$\frac{2A_1}{1.1} + \frac{6\times 5A_1}{(1.1)^5} = 4403 \quad > \quad 100\left(\frac{2\times 1}{1.1^2} + \frac{5\times 4}{1.1^4} + \frac{6\times 5}{1.1^6}\right) = 3225$

9.1.1.

(a) No-arbitrage forward price $S_0 e^{rT} = 2000 \times e^{5\% \times 1} = 2102.54$.

(b) ① Take the short position on one-year forward contract in price 2150.

② Borrow 2000 to buy one ounce

③ After one year, give the platinum to buyer, and payback $2000e^{0.05} = 2102.54$ to lender, get profit $2150 - 2102.54 = 47.46$.

(c) Since no arbitrage, portfolio A: long position + $Ke^{-r(T-t)}$ cash Portfolio B: 1 unit asset.
After T-t, value A is $K + f_{T,t}$, value B is $K + f_{T,t}$, then present value
$f_t = S_t - Ke^{-r(T-t)} = 2000 - 2102.54e^{-0.05\times 0.5} = -50.63$.

9.1.2.

(a) $P_1 = S_0 e^{rT} = 900e^{0.08} = 974.96$, $P_2 = 900e^{0.08\times 2} = 1056.16$.

(b) At time 1, 2-year long forward contract $S_1 - 900e^{1\times(0.08)}$
3-year short contract $900e^{3-2\times(0.08)} - S_1$, combine is 0.

(c) 2-year long $S_1 - 900e^{2(0.08)}e^{-0.10}$ 3-year short $900e^{3(0.05)}e^{-2(0.10)} - S_1$

Combined value is -18.92.